* 1. **Design of region *D5***
     1. **Generate an interpolation curve from the given points based on the Discrete Fourier Transform**

The PCHIP can only be applied to points that have a function relation between ys and xs, namely for every y, there is only one coresonding x. That is to say, it cannot be applied to parametric curves. That is why **Fourier Transform** is introduced. Consider an arbitrary closed (end-to-end) parametric curve . Denote it as a complex function

Note that this complex function can be expressed as a sum of infinite simple orbiting complex numbers with different frequencies and complex coefficients to describe the phases and amplitude. Hence, there is

This series, known as the **Fourier series**, was proposed by Joseph Fourier in 1807. The reason why the frequency is is that for any closed curve, is true if and only if ; you can see this as a sort of period ( ). The **Fourier Transform** is used to transform a function from the original domain to the frequency domain, namely to get corresponding s for different frequencies s, which is:

Proof: According to Eqs. 4 and 2,

According to **Euler’s formula** , the complex number really describes a “revolution”. Therefore, if integrate it through t, it will give zero as:

Hence, all terms in Eq. 15 vanish except . However, in the real-world problem, it is unlikely to get a continuous curve, and most of the time, we have to deal with discrete sample points. Suppose there are discrete points on a curve in complex form: . The **Discrete Fourier Transform** and the Fourier series fetched are:

A higher can give a more accurate fitting of the original curve, and a relatively small can filter the high-frequency details and give a gentler (no steep wiggle) shape.

* + 1. **Design with the Fourier series**

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| --- | --- | --- |
| Given an original graph like Figure 1, first take sample points on the contour of it. Note that the contour must be enclosed for the periodic property of the Fourier series. Since this is a rather complex shape, to keep more details, 2500 sample points are taken. Next, the only parameter that needs to be adjust is the number of terms retained from the series (the value of *M*). Different *M*s give different curves (see Figure 2). | Figure 1. D5 original graph | Figure 2. Sampled graph |

Table 1. Different values of M and corresponding curves

|  |  |  |
| --- | --- | --- |
| M=2 | M=10 | M=100 |
|  |  |  |
| M=200 | M=1000 | M=2500 |
|  |  |  |

The process to get the Discrete Fourier Transform from 2500 sample points requires massive calculation. Therefore, a python program is used. Notice that when M=2500, curve start to have sawteeth from the pixels of the original image, which is not expected. Hence M=1000 is considered suitable. Now an enclosed curve is obtained. It only consists of one curve. Thus it has continuity everywhere. Translate it from complex form to vector form: